Homework 8b

Math 253, Spring 2021

Name: Key

- This packet is due at **12:00 pm PDT (noon) on Saturday, May 29**.
- Neatly fill out this packet either directly or on your own paper. Then upload your answers to Canvas. Please clearly number each problem and keep them in order.
- For each problem in this packet, **organize and show all of your work and justification** for your answer.
- Make sure that it is clear what your final answer is. This may mean boxing/circling it or it may mean doing neither.
- If you are required to draw a graph of a function, each axis must be labeled and given a scale.
- I may not be grading every individual problem, but answers will be available on Canvas after the due date.
- I am happy to help you however I can. Here are some options:
  - Come to my office hours (see syllabus).
  - Email me directly at gknapp4@uoregon.edu

1. Find the Taylor polynomial of degree 2 for the function $f(x) = \cos(x)$ centered at $a = \pi$.

$f(\pi) = -1 \rightarrow c_0 = -1$

$f'(x) = -\sin(x) \rightarrow f'(\pi) = 0 \rightarrow c_1 = 0$

$f''(x) = -\cos(x) \rightarrow f''(\pi) = 1 \rightarrow c_2 = \frac{1}{2}$

$P_2(x) = -1 + \frac{1}{2}(x - \pi)^2$

2. Compute the Taylor series of $g(x) = \ln(x)$ centered at $a = 1$.

$g(1) = 0 \rightarrow c_0 = 0$

$g'(x) = \frac{1}{x} \rightarrow g'(1) = 1 \rightarrow c_1 = 1$

$g''(x) = \frac{-1}{x^2} \rightarrow g''(1) = -1 \rightarrow c_2 = -\frac{1}{2}$

$g'''(x) = \frac{2}{x^3} \rightarrow g'''(1) = 2 \rightarrow c_3 = \frac{2}{6} = \frac{1}{3}$

$g''''(x) = \frac{-6}{x^4} \rightarrow g''''(1) = -6 \rightarrow c_4 = \frac{-6}{24} = -\frac{1}{4}$

$\vdots$

$g^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \rightarrow g^{(n)}(1) = (-1)^{n+1}(n-1)! \rightarrow c_n = \frac{(-1)^{n+1}(n-1)!}{n!} = \frac{(-1)^{n+1}}{n}$

$\Rightarrow$ Taylor series for $\ln(x)$ centered at 1 is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$

3. Use the Maclaurin Series for $\cos(x)$ and $\sin(x)$ to show that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$. You don't need to derive the Maclaurin series for either function since we did $\cos(x)$ in class and you just did $\sin(x)$ in homework 8a.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\Rightarrow \cos(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (-x)^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left((-x)^2\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot (x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n}$$

$$= \cos(x)$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$$

$$\Rightarrow \sin(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (-x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (-1)^{2n+1} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot (-1)\, x^{2n+1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$$

$$= -\sin(x)$$

4. Find the smallest value of n such that the remainder estimate $|R_n| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1}$ for the *Maclaurin Series* for $f(x) = e^{-x}$ yields $|R_n| \leqslant 10^{-3}$ at $x = 3$. Compute $p_n(3)$ using your favorite computational tool.

$f^{(n)}(x) = (-1)^n e^{-x} \Rightarrow |f^{(n)}(x)| = e^{-x}$

$\Rightarrow |f^{(n)}(x)| \leq 1$ on $[0,3]$

$\Rightarrow |R_n| \leq \frac{1}{(n+1)!}(3-0)^{n+1} = \frac{3^{n+1}}{(n+1)!}$

Want: $\frac{3^{n+1}}{(n+1)!} < 10^{-3}$

Trial and Error: $n = 12$

$\Rightarrow e^{-3} \approx \sum_{n=0}^{12} \frac{(-1)^n \cdot 3^n}{n!} \approx 0.050$